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SYSTEMS OF ACOUSTIC RESONATORS IN THE QUASISTATIONARY MODE†

R. R. GADYL'SHIN

Ufa

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The method of matched asymptotic expansions is used to investigate the quasistationary mode of operation of a system of acoustic resonators with small apertures, which are either embedded in one another or are series-connected resonators. Asymptotic forms with respect to a small parameter (the "radius" of the aperture) of the poles, which converge to zero, are constructed for the analytic extension of Green's function of such systems. Peak values of the principal terms of the asymptotic forms of the solutions for the scattering and radiation problems are given.

The Helmholtz resonator is an ideally rigid "almost closed" surface $\Gamma_\varepsilon = \Gamma\setminus\overline{\omega}_\varepsilon$, where Γ is the boundary of the finite volume Ω, while ω_ε is the part of Γ of "radius" $\varepsilon^2 \ll 1$ [1–5]. The scattering of an external field with potential velocity $\mathbf{v}^{\text{out}} = \nabla u^{\text{out}}$ by Γ_ε. is described by the solution of a Neumann boundary-value problem for the Helmholtz equation with boundary condition on Γ_ε. The resonance phenomena consist, in particular, of the fact that at frequencies k close to the natural frequencies of the volume Ω, the field scattered by Γ_ε differs considerably from the field scattered by Ω [1–3, 5]. The resonances are explained as follows [6]. Green's function of the internal limiting problem (Neumann's problem in Ω) in the neighbourhood of the simple eigenvalue k_0^2 has the form

$$G(\mathbf{x},\mathbf{y},k) = (k_0^2 - k^2)^{-1}\psi(\mathbf{x})\psi(\mathbf{y}) + g(\mathbf{x},\mathbf{y},k)$$

where ψ is the corresponding eigenfunction, while the function $g(\mathbf{x}, \mathbf{y}, k)$ is regular with respect to the variable k. The resonances are a consequence of the existence in Green's function $G_\varepsilon(\mathbf{x}, \mathbf{y}, k)$ of the resonator of complex poles which, as the aperture is reduced, converge to real poles (the natural frequencies) k_0 of Green's function of the limiting problem. If $k_0 \neq 0$, then $G(\mathbf{x}, \mathbf{y}, k)$ has a pole of the first order in k. The pole of the function $G_\varepsilon(\mathbf{x}, \mathbf{y}, k)$ which converges to k_0, inherits the same order. If $k_0 = 0$, then $G(\mathbf{x}, \mathbf{y}, k)$ has a pole of the second order (in k), while the function $G_\varepsilon(\mathbf{x}, \mathbf{y}, k)$ has two poles of the first order, which tend to zero, and the residues at these poles increase without limit as $\varepsilon \to 0$. The latter fact distinguishes the quasistationary mode of operation from the other resonance modes [1, 8].

We also know [6, 9], that if k_0^2 is a double eigenvalue of the closed resonator, the perturbed problem will have two poles which converge to k_0. The peaks of the solutions corresponding to one of these are considerably greater than the peaks corresponding to simple eigenvalues. On the other hand, we know [10], that if the resonator for which $k_0 \neq 0$ is a simple natural frequency of the limiting problem, is a surrounded by a resonator of the spherical-layer type, for which k_0 is not a natural limiting frequency, then, nevertheless, the resonance peaks in the "internal" volume are increased. Hence, an unusual boosting of the peaks by the "non-resonant" volume occurs.

In this paper we investigate a system of two resonators embedded in one another (and also a chain of resonators [11]), under quasistationary conditions. Taking the above into account, it will be natural to

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expect that in this case the resonance peaks should be increased compared with the quasistationary mode for a single resonator. However, as will be shown below, no increase in the resonance peaks is observed. The difference is that a second pair of peaks (poles) of the same order appears.

1. FORMULATION OF THE PROBLEM AND PRELIMINARY DATA

Suppose Ω_0 and Ω are bounded simply connected regions in $\mathbf{R}^3$, $\overline{\Omega}_0 \subset \Omega$, their boundaries $\Gamma_0^{(0)} = \partial\Omega_0$, $\Gamma_0^{(1)} = \partial\Omega \in C^\infty$ are flattened in the neighbourhood of the points $\mathbf{x}_0^{(m)} \in \Gamma_0^{(m)}$, $\Gamma_\varepsilon^{(m)} = \Gamma_0^{(m)} \setminus \overline{\omega}_\varepsilon^{(m)}$, $\omega_\varepsilon^{(m)} = \{\mathbf{x} : (\mathbf{x} - \mathbf{x}_0^{(m)})\varepsilon^{-2} \in \omega^{(m)}\}$, $\omega^{(m)}$ are two-dimensional simply connected regions in the $T^{(m)}$ planes, which coincide in the neighbourhood of $\mathbf{x}_0^{(m)}$ with $\Gamma_0^{(m)}$, and the boundaries $\partial\omega^{(m)} \in C^\infty$. We will put $\Gamma_\delta = \Gamma_\delta^{(0)} \cup \Gamma_\delta^{(1)}$, $\Omega_1 = \Omega \setminus \overline{\Omega}_0$, $\Omega_2 = \mathbf{R}^3 \setminus (\Omega_0 \cup \Omega_1)$. In this notation the boundary-value problem for a system of embedded resonators has the form

$$(\Delta + k^2)u_\varepsilon = 0, \ \mathbf{x} \in \mathbf{R}^3 \setminus \overline{\Gamma}_\varepsilon, \ \partial u_\varepsilon / \partial \mathbf{n} = f, \ \mathbf{x} \in \Gamma_\varepsilon; \ f \in C^\infty(\Gamma_0) \tag{1.1}$$

$$\partial u_\varepsilon / \partial r - iku_\varepsilon = o(r^{-1}), \ r \to \infty; \ r = |\mathbf{x}| \tag{1.2}$$

($\mathbf{n}$ is the outward normal). For the scattering problem $f = \partial u^{\text{out}} / \partial n$. The solution of boundary-value problem (1.1), (1.2) is considered in the class of functions belonging to $W_2^1(S(R) \setminus \overline{\Gamma}_\varepsilon)$ for any R, and $S(R)$ is a sphere of radius R with centre at the origin of coordinates. The surfaces $\Gamma_\varepsilon^{(m)}$ are understood as being two-sided.

If $\Omega \cap \Omega_0 = \varnothing$ and the boundary $\Gamma_0^{(1)}$ of the region $\Omega_1 = \Omega$ in the neighbourhood of $\mathbf{x}_0^{(0)}$ coincides with $\Gamma_0^{(0)}$ and $\mathbf{x}_0^{(1)} \notin \Gamma_0^{(0)}$, then (1.1), (1.2) describes a boundary-value problem for a chain of series-connected resonators $\Gamma_\varepsilon^{(j)}$ [11]. The asymptotic forms will be constructed for both systems without any differences. In this case Ω_1 is understood, in a corresponding way, to be the defined region.

The residue of the analytic extension of $G_\varepsilon(\mathbf{x}, \mathbf{y}, k)$ at the pole τ_ε is the solution of the boundary-value problem (1.1) when $k = \tau_\varepsilon$, $f = 0$. Naturally, for fixed ε it increases exponentially as $r \to \infty$. By analogy with problems in bounded regions we will call the solutions of the homogeneous boundary-value problems eigenfunctions.

The assertions which enable us to justify the above asymptotic constructions, and also formulae (1.3)–(1.5) for the solutions of problem (1.1), (1.2), for systems of embedded resonators are proved in [6, 7]. Similar results for a chain of resonators can be obtained using the techniques described in [12]. When $j = 0, 1$ we will denote by ψ_j functions equal to $\text{mes}^{-1/2}\Omega_j$ in $\overline{\Omega}_j$ and zero outside $\overline{\Omega}_j$. If there are two poles $\tau_\varepsilon^{(1)}$, $\tau_\varepsilon^{(2)} \to 0$ in the system of resonators when $\varepsilon \to 0$ such that $0 < \text{Re}\,\tau_\varepsilon^{(1)} < \text{Re}\,\tau_\varepsilon^{(2)}$, then there are exactly two poles

$$\tau_\varepsilon^{(3)} = -\overline{\tau_\varepsilon^{(1)}}, \ \tau_\varepsilon^{(4)} = -\overline{\tau_\varepsilon^{(2)}} \tag{1.3}$$

and the orders of all the poles $\tau_\varepsilon^{(q)}$ are equal to unity. If, moreover, $\text{Im}\,\tau_\varepsilon^{(q)} = o(\text{Re}\,\tau_\varepsilon^{(q)})$ as $\varepsilon \to 0$, while the corresponding eigenfunctions $\Phi_\varepsilon^{(q)}$, normalized in $L_2(\Omega_0 \cup \Omega_1)$, the following relation holds

$$\lim_{\varepsilon \to 0} \int_{\Omega_0 \cup \Omega_1} \Phi_\varepsilon^{(1)} \overline{\Phi_\varepsilon^{(2)}} d\mathbf{x} = 0 \tag{1.4}$$

then for k close to zero the following representation holds for the solution of boundary-value problem (1.1), (1.2)

$$u_\varepsilon(\mathbf{x}, k) = \sum_{n=1}^{2} (2\text{Re}\tau_\varepsilon^{(n)})^{-1} \left(\frac{\psi_\varepsilon^{(n)}(\mathbf{x})}{\tau_\varepsilon^{(n)} - k} \sum_{m=0}^{1} \int_{\Gamma_0^{(m)}} \{\psi_\varepsilon^{(n)}\} f_m ds + \frac{\overline{\psi_\varepsilon^{(n)}(\mathbf{x})}}{\overline{\tau_\varepsilon^{(n)}} + k} \sum_{m=0}^{1} \int_{\Gamma_0^{(m)}} \left\{ \overline{\psi_\varepsilon^{(n)}} \right\} f_m ds \right) + \\ + U_\varepsilon(x, k) \tag{1.5}$$

where f_m is the value of the function f on $\Gamma_0^{(m)}$. As $\varepsilon \to 0$ the eigenfunctions $\psi_\varepsilon^{(n)} \to \psi_n$ in $L_2(K)$ for any compactum $K \subset \mathbf{R}^3$, $\Psi_n = \alpha_{n0}\psi_0 + \alpha_{n1}\psi_1$, $\alpha_{n0}^2 + \alpha_{n1}^2 = 1$, $\alpha_{10}\alpha_{20} + \alpha_{11} + \alpha_{21} = 0$. The function U_ε is uniformly bounded in the same norm, converges to the solution of Neumann's problem in Ω_2 with respect to the norm $L_2(\Omega_2 \cap K)$ and

$$\sum_{m=0}^{1} \int_{\Gamma_0^{(m)}} \{\Psi_\varepsilon^{(n)}\} f_m ds \to a_f^{(n)} = (\alpha_{n0}\psi_0 - \alpha_{n1}\psi_1) \int_{\Gamma_0^{(0)}} f_0 ds + \alpha_{n1}\psi_1 \int_{\Gamma_0^{(1)}} f_1 ds \tag{1.6}$$

The symbol $\{\cdot\}$ denotes a discontinuity of the function on the surface $\Gamma_0^{(m)}$.

2. FORMULATION OF THE FUNDAMENTAL ASSERTIONS

The principal terms of the asymptotic forms of the poles $\tau_\varepsilon^{(q)}$ of Green's function of problem (1.1), (1.2) on the corresponding eigenfunctions depend on certain characteristics of the closed resonators and the openings. We will introduce the following notation: $G_j(\mathbf{x}, \mathbf{y}, k)$ is Green's function of Neumann's problem in Ω_j, $\sigma(k) = \lim_{R\to\infty} \int_{r=R} |G_2(\mathbf{x}, \mathbf{x}_0^{(1)}, k)|^2 ds$ is the scattering cross section [2, 13], and $c(\omega)$ is the capacity of the disc ω [14, 15]. The constants $\sigma(k)$, $c(\omega) > 0$, while if ω is the unit circle, then $c(\omega) = 2\pi^{-1}$ [15]. Note that if $\operatorname{Im} k = o(\operatorname{Re} k)$ as $k \to 0$, then [8]

$$\operatorname{Im} G_2(\mathbf{x}_0^{(1)}, \mathbf{x}_0^{(1)}, k) = \sigma \ \operatorname{Re} k + o(k), \quad \sigma = \sigma(0)$$

Suppose $\xi = (\xi_1, \xi_2, \xi_3)$ and ω is a two-dimensional bounded region in the plane $\xi_3 = 0$. We will denote by $Y_0(\xi; \omega)$ a function which is harmonic outside ω, which falls off at infinity, which belongs to $W_{2,loc}^1(\mathbf{R}^3 \setminus \overline{\omega})$ and is identically equal to unity on ω. We will put $Y(\xi; \omega) = 1 - 1/2Y(\xi; \omega)$ when $\xi_3 \geqslant 0$ and $Y(\xi; \omega) = 1/2Y_0(\xi; \omega)$ when $\xi_3 \leqslant 0$. The following lemma is verified directly.

Lemma 1. The system of equations

$$(R_0^{(0)}\psi_0)^2 + (R_0^{(1)} - R_0^{(0)})^2 \psi_1^2 = 1 \tag{2.1}$$

$$c(\omega^{(0)})(R_0^{(0)}(\psi_0^2 + \psi_1^2) - R_0^{(1)}\psi_1^2) = R_0^{(0)}\tau_1^2\pi^{-1} \tag{2.2}$$

$$c(\omega^{(1)})\psi_1^2(R_0^{(1)} - R_0^{(0)}) = R_0^{(1)}\tau_1^2\pi^{-1} \tag{2.3}$$

in τ_1, $R_0^{(0)}$, $R_0^{(1)}$ has the following two sets of real solutions ($n = 1, 2$)

$$\tau_1 = \tau_1^{(n)} = (\tfrac{1}{2}\pi(\zeta - (-1)^n(\zeta^2 - 4c(\omega^{(0)})c(\omega^{(1)})\psi_0^2\psi_1^2)^{1/2})^{1/2}$$

$$\zeta = c(\omega^{(0)})(\psi_0^2 + \psi_1^2) + c(\omega^{(1)})\psi_1^2$$

$$R_0^{(1)} = R_0^{(1,n,0)} = c(\omega^{(1)})\psi_1^2((\psi_0^2 + \psi_1^2)\pi^{-2}(\tau_1^{(n)})^4 -$$

$$-2c(\omega^{(1)})\psi_0^2\psi_1^2\pi^{-1}(\tau_1^{(n)})^2 + c^2(\omega^{(1)})\psi_0^2\psi_1^4\pi^{-2}(\tau_1^{(n)})^4)^{-1/2}.$$

$$R_0^{(0)} = R_0^{(0,n,0)} = (1 - (c(\omega^{(1)})\psi_1^2\pi)^{-1}(\tau_1^{(n)})^2)R_0^{(1,n,0)}$$

Corollary. Constants $\tau_1^{(1)} > \tau_1^{(2)} > 0$, $R_0^{(m,n,0)} \neq 0$, $R_0^{(1,n,0)} \neq R_0^{(0,n,0)}$, $R_0^{(0,1,0)}R_0^{(1,2,0)} \neq R_0^{(1,1,0)}R_0^{(0,2,0)}$, $R_0^{(m,1,0)} \neq R_0^{(m,2,0)}$, where

$$R_0^{(0,1,0)}R_0^{(0,2,0)}\psi_0^2 + (R_0^{(1,1,0)} - R_0^{(0,1,0)})(R_0^{(1,2,0)} - R_0^{(0,2,0)})\psi_1^2 = 0$$

Suppose $\mathbf{x}_m = (x_1^{(m)},\ x_2^{(m)},\ x_3^{(m)})$ is a system of coordinates obtained from $\mathbf{x}$ by an orthogonal transformation such that in this system $\mathbf{x}_0^{(m)}$ coincides with the origin of coordinates, while the region Ω_m in the neighbourhood of $\mathbf{x}_0^{(m)}$ coincides with the half-space $x_3^{(m)} > 0$. We will denote by $S_m(R)$ a sphere of radius R with centre at $\mathbf{x}_0^{(m)}$. The main content of this paper is the following assertion, the proof of which is based on the method of matched asymptotic expansions [16–18] and will be given below.

Theorem 1. Four first-order poles $\tau_\varepsilon^{(n)}$ of Green's function of boundary-value problem (1.1), (1.2) exists, connected by Eqs (1.3). The asymptotic forms of these poles and the corresponding eigenfunctions for $n = 1, 2$ have the following form

$$\tau_\varepsilon^{(n)} = \varepsilon\tau_1^{(n)} + \varepsilon^3\tau_3^{(n)} + \varepsilon^4\tau_4^{(n)} + O(\varepsilon^5)$$

$$\Psi_\varepsilon^{(n)}(\mathbf{x}) \sim R_0^{(0,n,0)}\psi_0^2 \ \text{ for } \mathbf{x} \in \Omega_0 \setminus S_0(\varepsilon)$$

$$\Psi_\varepsilon^{(n)}(\mathbf{x}) \sim (R_0^{(1,n,0)} - R_0^{(0,n,0)})\psi_1^2 \ \text{ for } \mathbf{x} \in \Omega_1 \setminus (S_0(\varepsilon) \cup S_1(\varepsilon))$$

$$\Psi_\varepsilon^{(n)}(\mathbf{x}) \sim \varepsilon^2(\tau_1^{(n)})^2 R_0^{(1,n,0)} G_2(\mathbf{x}, \mathbf{x}_0^{(1)}, \tau_\varepsilon^{(n)}) \ \text{ for } \mathbf{x} \in \Omega_2 \setminus S_1(\varepsilon)$$

$$\Psi_\varepsilon^{(n)}(\mathbf{x}) \sim \upsilon_0^{(m,n)}(\mathbf{x}_m / \varepsilon^2) \ \text{ for } \mathbf{x} \in S_m(2\varepsilon),\ m = 0,1$$

in norms $L_2(K)$ for any compactum $K \subset \mathbf{R}^3$. The constants $\tau_1^{(n)}$, $R_0^{(0,n,0)}$, $R_0^{(1,n,0)}$ are the solutions of system (2.1)–(2.3) and

$$\operatorname{Im}\tau_3^{(n)} = 0,\ \ \operatorname{Im}\tau_4^{(n)} = -\tfrac{1}{2}\pi(\tau_1^{(n)})^2 c(\omega^{(1)})\sigma \tag{2.4}$$

$$\upsilon_0^{(0,n)}(\xi) = R_0^{(0,n,0)}\psi_0^2 Y(\xi;\omega^{(0)}) + (R_0^{(1,n,0)} - \tag{2.5}$$

$$-R_0^{(0,n,0)})\psi_1^2 Y(\xi_*;\omega^{(0)}),\ \ \xi_* = (\xi_1, \xi_2,\ \xi_3)$$

$$\upsilon_0^{(1,n)}(\xi) = (R_0^{(1,n,0)} - R_0^{(0,n,0)})\psi_1^2 Y(\xi;\omega^{(1)}) \tag{2.6}$$

Corollary. Relation (1.4) holds.

The correctness of the last assertion follows from the asymptotic forms of the principal terms $\Psi_\varepsilon^{(n)}$ in Ω_m and the corollary of Lemma 1.

It follows from (1.5) and Theorem 1 that for real $k = k(\varepsilon)$ the solution of boundary-value problem (1.1), (1.2) will experience the greatest perturbations in the peak modes

$$k_\pm = \pm(\varepsilon\tau_1^{(n)} + \varepsilon^3\tau_3^{(n)} + \varepsilon^4(t + o(1)))$$

where t is an arbitrary real number. The following theorem follows directly from (1.5), (1.6) and Theorem 1 for the radiation problem ($a_f^{(n)} \neq 0$).

Theorem 2. In the peak mode k_+ the solution of boundary-value problem (1.1), (1.2) has the asymptotic forms

$$u_\varepsilon(x,k) \sim \varepsilon^{-5}T\ a_f^{(n)} R_0^{(0,n,0)}\psi_0^2 \ \text{ for } \mathbf{x} \in \Omega_0 \setminus S_0(\varepsilon)$$

$$u_\varepsilon(x,k) \sim \varepsilon^{-5}T\ a_f^{(n)}(R_0^{(1,n,0)} - R_0^{(0,n,0)})\psi_1^2 \ \text{ for } \mathbf{x} \in \Omega_1 \setminus S_0(\varepsilon) \cup S_1(\varepsilon))$$

$$u_\varepsilon(x,k) \sim \varepsilon^{-5}T\ a_f^{(n)}\upsilon_0^{(m,n)}\mathbf{x}_m / \varepsilon^2) \ \text{ for } \mathbf{x} \in S_m(2\varepsilon),\ m = 0,1$$

$$u_\varepsilon(x,k) \sim \varepsilon^{-3}T(\tau_1^{(n)})^2 a_f^{(n)} R_0^{(1,n,0)} G_2(\mathbf{x}, \mathbf{x}_0^{(1)}, k) \ \text{ for } \mathbf{x} \in \Omega_2 \setminus S_1(\varepsilon)$$

in the norm $L_2(K)$; $T = (2\tau_1^{(n)}(\tau_4^{(n)} - t))^{-1}$.

Consider the scattering problem. Suppose $u^{\text{out}}(\mathbf{x},\ k)$ is a certain external field, and $u_0(\mathbf{x},\ k)$ the field scattered by an ideally rigid body $\Omega_0 \cup \Omega_1$ (the solution of Neumann's problem in Ω_2 with boundary condition $f = \partial u^{\text{out}} / \partial\mathbf{n}$), while $u(\mathbf{x},\ k) = u_0(\mathbf{x},\ k) + u^{\text{out}}(\mathbf{x},\ k)$ is the total field. It is obvious that in this case $a_f^{(n)} = 0$. However, using the asymptotic form of the eigenfunctions in

Ω_2 it can be shown (see, for example, [6]), that in peak modes

$$\sum_{j=0}^{1}\int_{\Gamma_0^{(j)}}\left\{\psi_\varepsilon^{(n)}\right\}f_j ds=\varepsilon^2\left((\tau_1^{(n)})^2 R_0^{(1,n,0)}u(\mathbf{0},0)+o(1)\right) \tag{2.7}$$

The following theorem follows from (1.5), (2.7) and Theorem 1.

Theorem 3. In the peak mode k_+ the scattered field has the following asymptotic forms

$$u_\varepsilon(\mathbf{x},k)\sim\varepsilon^{-3}Tb_f^{(n)}R_0^{(0,n,0)}\psi_0^2 \text{ for } \mathbf{x}\in\Omega_0\setminus S_0(\varepsilon)$$

$$u_\varepsilon(\mathbf{x},k)\sim\varepsilon^{-3}Tb_f^{(n)}(R_0^{(1,n,0)}-R_0^{(0,n,0)})\psi_1^2 \text{ for } \mathbf{x}\in\Omega_1\setminus S_0(\varepsilon)\cup S_1(\varepsilon))$$

$$u_\varepsilon(\mathbf{x},k)\sim\varepsilon^{-3}Tb_f^{(n)}\upsilon_0^{(m,n)}(\mathbf{x}_m/\varepsilon^2) \text{ for } \mathbf{x}\in S_m(2\varepsilon),\ m=0,1$$

$$u_\varepsilon(\mathbf{x},k)\sim\varepsilon^{-1}T(\tau_1^{(n)})^2 b_f^{(n)}R_0^{(1,n,0)}G_2(\mathbf{x},\mathbf{x}_0^{(1)},k) \text{ for } \mathbf{x}\in\Omega_2\setminus S_1(\varepsilon)$$

in $L_2(K)$; $b_f^{(n)}=(\tau_1^{(n)})^2R_0^{(1,n,0)}$.

3. CONSTRUCTION OF THE ASYMPTOTIC FORMS

The complete asymptotic forms of the poles $\tau_\varepsilon^{(n)}$ and the corresponding eigenfunctions will be sought in the form

$$\tau_\varepsilon^{(n)}=\sum_{i=1}^{\infty}\varepsilon^i\tau_i^{(n)} \tag{3.1}$$

$$\psi_\varepsilon^{(n)}(\mathbf{x})=-k^2\sum_{i=0}^{\infty}\varepsilon^i R_{[i/2]}^{(0,n,i)}G_0(\mathbf{x},\mathbf{x}_0^{(0)},k),\ \mathbf{x}\in\Omega_0\setminus S_0(\varepsilon) \tag{3.2}$$

$$\psi_\varepsilon^{(n)}(\mathbf{x})=k^2\left(\sum_{m=0}^{1}\sum_{i=0}^{\infty}(-1)^m\varepsilon^i R_{[i/2]}^{(m,n,i)}G_1(\mathbf{x},\mathbf{x}_0^{(m)}k)\right),\quad \mathbf{x}\in\Omega_1\setminus S_0(\varepsilon)\cup S_1(\varepsilon) \tag{3.3}$$

$$\psi_\varepsilon^{(n)}(\mathbf{x})=k^2\sum_{i=0}^{\infty}\varepsilon^i R_{[i/2]}^{(i,n,i)}G_2(\mathbf{x},\mathbf{x}_0^{(1)},k),\ \mathbf{x}\in\Omega_2\setminus S_1(\varepsilon) \tag{3.4}$$

$$\psi_\varepsilon^{(n)}(\mathbf{x})=\sum_{i=0}^{\infty}\varepsilon^i\upsilon_i^{(m,n)}(\mathbf{x}_m/\varepsilon^2),\ \mathbf{x}\in S_m(2\varepsilon),\ m=0,1 \tag{3.5}$$

where $k=\tau_\varepsilon$, $n=1,2$, $R_j^{(q,i,s)}(D_y)$ are differential polynomials of degree j in the variable $\mathbf{y}=(y_1, y_2, y_3)$ with constant coefficients, and $[N]$ is the integer part of N. Note that the eigenfunctions Ψ_ε and ψ_ε which occur in (1.5), (1.6) and (3.2)–(3.5), differ from one another by the factor $1+o(1)$.

The boundary-value problems for the coefficients of series (3.5) are obtained in the following way, which is standard for the method of matched asymptotic expansions [6, 8, 17]. In (1.1) we assume $f=0$, and instead of k and u_ε we substitute the series (3.1) and (3.5), respectively, and change in (1.1) to the variable $\xi=\mathbf{x}_m\varepsilon^{-2}$. We then write the equations separately for similar powers of ε and pass to the formal limit as $\varepsilon\to 0$. We finally obtain the following system of boundary-value problems

$$\Delta_\xi\upsilon_j^{(m,n)}=-\sum_{i=2}^{j-2}\lambda_i^{(n)}\upsilon_{j-i-2}^{(m,n)},\ \xi\in\bar{\gamma}^{(m)} \tag{3.6}$$

$$\partial\upsilon_j^{(m,n)}/\partial\xi_3=0,\ \xi\in\gamma^{(m)},\ \gamma^{(m)}=\left\{\xi=(\xi_1,\xi_2,0):\ \xi\notin\overline{\omega}^{(m)}\right\}$$

where λ_i are the coefficients of the series $\lambda_\varepsilon^{(n)}=(\tau_\varepsilon^{(n)})^2$.

We will seek the differential polynomials $R_j^{(m,n,i)}$ in the form

$$R_j^{(m,n,i)}(D_y) = \sum_{q=0}^{j} P_q^{(m,n,i,j)}(D_y) \tag{3.7}$$

$$P_q^{(m,n,i,j)}(D_y) = \sum_{t=0}^{j} a_{qt}^{(m,n,i,j)} \frac{\partial^j}{\partial^t y_1^{(m)} \partial^{q-t} y_2^{(m)}}, \quad \mathbf{y}_m = (y_1^{(m)},\ y_2^{(m)},\ y_3^{(m)})$$

where $a_{ji}^{(\cdot)}$ are certain constants. For $q = 0, 1, 2$ we will denote the series (3.2)–(3.4) by $\psi_\varepsilon^{(q,n)}(\mathbf{x}, k)$, respectively, where k is not replaced by $\tau_\varepsilon^{(n)}$. The asymptotic forms (3.2)–(3.4) for the eigenfunction $\psi_\varepsilon(\mathbf{x})$ then have the form $\psi_\varepsilon^{(q,n)}(\mathbf{x},\ \tau_\varepsilon^{(n)})$. By virtue of the definition of the differential polynomials the coefficients of the series $\psi_\varepsilon^{(q,n)}(\mathbf{x}, k)$ are analytic in a certain neighbourhood of zero (with respect to k), satisfy the homogeneous and Neumann boundary condition on $\Gamma_0 \setminus \{\cup \mathbf{x}_0^{(m)}\}$, and are solutions of the Helmholtz equation in Ω_q, while for real k the coefficients of the series $\psi_\varepsilon^{(2,n)}(\mathbf{x}, k)$ also satisfy condition (1.2).

For Green's functions $G_t(\mathbf{x}, \mathbf{y}, k)$ and their derivatives in the neighbourhood of the points $\mathbf{x}_0^{(m)}$ the following representations hold $(s = 0, 1)$

$$P_q^{(m,n,i,j)}(D_y) G_{m+s}(\mathbf{x}, \mathbf{x}_0^{(m)}, k) = (2\pi)^{-1}(-1)^q P_q^{(m,n,i,j)}(D_x)\left(r^{-1} e_m^{ikr}\right) +$$
$$+ g_q^{(m,n,i,j)}(\mathbf{x}, k) - k^{-2} \delta_0^q (1 - \delta_2^{m+s}) P_q^{(m,n,i,j)} \psi_{m+s}^2$$

where δ_q^t is the Kronecker delta, $r_m = |\mathbf{x} - \mathbf{x}_0^{(m)}|$, and the functions $g_q^{(\cdot)}(\mathbf{x}, k)$ in the neighbourhood of $\mathbf{x}_0^{(m)}$ are infinitely differentiable, satisfy the boundary condition $\partial g_q^{(\cdot)} / \partial x_3^{(m)} = 0$ for $x_3^{(m)} = 0$, and in a certain neighbourhood of zero are analytic in k. If the coefficients of the polynomials $P_j^{(\cdot)}$ and k are real, the functions $P_j^{(\cdot)}(D_y) G_g(\mathbf{x},\ \mathbf{x}_0^{(m)},\ k)$ are also real for $s = 0, 1$.

Suppose $T_j(\mathbf{x})$ are homogeneous functions of degree j, which are either homogeneous polynomials or the product of homogeneous polynomials in r^{-2q-1} for certain integer $q \geqslant 0$, and satisfy the boundary condition $\partial T_j(\mathbf{x}) / \partial x_3 = 0$ for $x_3 = 0$, $\mathbf{x} \neq 0$. We will denote by $\tilde{A}_j$ the set of series of the form

$$T(\mathbf{x}) = \sum_{q=-\infty}^{j} T_q(\mathbf{x})$$

We will say that two series are conjugate if their sum is a polynomial.

In the sums $U(\mathbf{x}, \varepsilon)$ of the form $\psi_\varepsilon^{(q,n)}(\mathbf{x},\ \tau_\varepsilon^{(n)})$ we define the operator $K_N^{(m)}$ as follows [6, 17]. We will expand the coefficients of the series $U(\mathbf{x}, \varepsilon)$ in series as $\mathbf{x}_m \to 0$ and change to the variable $\xi = \mathbf{x}_m \varepsilon^{-2}$. In the double series obtained we take the sum of terms of the form $\varepsilon^j \Phi(\xi)$ for $j \leqslant N$, which we will also call $K_N^{(m)}(U(\mathbf{x},\ \varepsilon))$. The following lemma follows from the asymptotic forms of Green's functions, the definition of the series $\psi_\varepsilon^{(q,n)}(\mathbf{x},\ \tau_\varepsilon^{(n)})$ and of the operator $K_N^{(m)}$.

Lemma 2. Suppose the arbitrary functions $\tau_\varepsilon^{(n)}$, the series $\psi_\varepsilon^{(q,n)}(\mathbf{x}, k)$ and the differential polynomials $R_j^{(m,n,i)}$ have the form (3.1)–(3.4) and (3.7), while in the representation (3.7) of the polynomials $R_j^{(1,n,i)}$ for $i \geqslant 1$ the coefficients $P_0^{(1,n,i,j)} = 0$. Then the following equalities hold for any integer $N \geqslant 0$

$$K_N^{(m)}(\psi_\varepsilon^{(m+s,n)}(\mathbf{x}, \tau_\varepsilon^{(n)})) = \sum_{i=0}^{N} \varepsilon^i V_i^{(m,m+s,n)}(\xi), \quad m, s = 0, 1$$

The series $V_i^{(m,m+s,n)}(\xi) \in \tilde{\mathbf{A}}_{[j/2]-1}$ are conjugate in pairs for any fixed $j \geqslant 2$, m and n, and are formal asymptotic solutions of boundary-value problem (3.6) as $\rho = |\xi| \to \infty$, where the functions $\upsilon_j^{(m,n)}$ are replaced by $V_j^{(m,m,+s,n)}$ and are represented in the form

$$V_0^{(m,m+s,n)}(\xi) = \tilde{V}_0^{(m,m+s,n)} - (-1)^{m+s} (2\pi)^{-1} \left(\tau_i^{(n)}\right)^2 \Big(R_0^{(m,n,0)} \rho^{-1} +$$
$$+ \sum_{i=1}^{\infty} (-1)^i P_i^{(m,n,2i,i)}(D_\xi) \rho^{-1} \Big)$$

$$V_j^{(m,m+s,n)}(\xi) = \tilde{V}_j^{(m,m+s,n)}(\xi) + 2\left(\tau_1^{(n)}\right)^{-1}\tau_{j+1}^{(n)}\left(V_0^{(m,m+s,n)}(\xi) - \tilde{V}_0^{(m,m+s,n)}\right) -$$

$$-(-1)^{m+s}(2\pi)^{-1}\left(\tau_1^{(n)}\right)^2 \sum_{i=m}^{\infty}(-1)^i P_i^{(m,n,2i+j,i)}(D_\xi)\rho^{-1}$$

$$\tilde{V}_0^{(0,0,n)} = R_0^{(0,n,0)}\psi_0^2, \quad \tilde{V}_0^{(m,1,n)} = \left(R_0^{(1,n,0)} - R_0^{(0,n,0)}\right)\psi_1^2$$

$$\tilde{V}_0^{(1,2,n)} = 0, \quad \tilde{V}_1^{(m,m+s,n)}(\xi) \equiv 0$$

where the series $\tilde{V}_j^{(m,m+s,n)}$ are independent of $\tau_{q+1}^{(n)}$ and $P_i^{(m,n,2i+q,i)}$ when $q \geqslant j$.
If moreover $\operatorname{Im}\tau_1^{(n)} = \operatorname{Im}R_0^{(m,n,0)} = P_1^{(m,n,2i+1,i)} = \tau_2^{(n)} = 0$, then

$$\operatorname{Im}\tilde{V}_3^{(1,2,n)}(\xi) = \left(\tau_1^{(n)}\right)^3 R_0^{(1,n,0)}\sigma, \quad \operatorname{Im}\tilde{V}_3^{(m,m,n)} = \operatorname{Im}\,\tilde{V}_3^{(0,1,n)} \equiv 0$$

We will denote by $\mathbf{A}_q^{(m)}$ the set of functions υ belonging to $W_1^2(S(R)\setminus\overline{\gamma^{(m)}})$ for any R and such that the sums $\upsilon(\xi)+\upsilon(\xi_*)$ are the polynomials of the qth order, while the asymptotic forms of the functions $\upsilon(\xi)$ as $\rho\to\infty$, $\pm\xi_3 \geqslant 0$ belong to $\tilde{\mathbf{A}}_q$. It follows from the definition of the classes $\mathbf{A}_q^{(m)}$ that the asymptotic forms of the function $\upsilon \in \mathbf{A}_q^{(m)}$ at infinity are conjugate. Note that the function $Y(\xi, \omega^{(m)}) \in \mathbf{A}_0^{(m)}$. The following lemma holds in this notation [6, 8].

Lemma 3. Suppose the function $F \in \mathbf{A}_q^{(m)}$, and the conjugate series $\tilde{V}^{in,ex} \in \tilde{\mathbf{A}}_{q+2}$ are formal asymptotic solutions of Neumann's problems in the half-spaces $\xi_3 \gtrless 0$ for the equations $\Delta\tilde{V}^{in,ex} = F$ as $\rho\to\infty$. A function $\upsilon \in \mathbf{A}_{q+2}^{(m)}$ then exists which is a solution of the boundary-value problem

$$\Delta\upsilon = F \text{ outside } \overline{\gamma^{(m)}}, \quad \partial\upsilon/\partial\xi_3 = 0 \text{ on } \gamma^{(m)}$$

and which, as $\rho\to\infty$, has the asymptotic forms

$$\upsilon(\xi) = \tilde{V}^{in,ex}(\xi) \pm \sum_{j=0}^{\infty} Z_j^{(m)}(\xi)\rho^{-2j-1}, \quad \xi_3 \lessgtr 0$$

where $Z_j^{(m)}$ are homogeneous harmonic polynomials of degree j, such that $\partial Z_j^{(m)}/\partial\xi_3 = 0$ when $\xi_3 = 0$.

Lemmas 2 and 3 enable us to match the asymptotic expansions (3.2)–(3.4). We will denote the partial sums of series (3.5) by $\upsilon_{\varepsilon,N}^{(m,n)}(\mathbf{x}_m/\varepsilon^2)$.

Theorem 4. Functions $\tau_\varepsilon^{(n)}$ and series $\psi_\varepsilon^{(q,n)}(\mathbf{x}, k)$ exist, which have the form (3.1)–(3.5), such that

(a) the coefficients $\upsilon_j^{(m,n)}(\xi) \in \mathbf{A}_{[j/2]-1}^{(m)}$ are solutions of boundary-value problems (3.6);

(b) for the coefficients of series (3.1)–(3.5) there are formulae (2.4)–(2.6), and the constants $\tau_1^{(n)}$, $R_0^{(0,n,0)}$, $R_0^{(1,n,0)}$ are solutions of system (2.1)–(2.3), $\tau_2^{(n)} = 0$;

(c) for any integer $N \geqslant 0$ as $\rho\to\infty$ the following equalities hold

$$\upsilon_{\varepsilon,N}^{(m,n)}(\xi) = K_N^{(m)}(\psi_\varepsilon^{(m,n)}(\mathbf{x}, \tau_\varepsilon^{(n)})), \quad \xi_3 \geqslant 0 \tag{3.8}$$

$$\upsilon_{\varepsilon,N}^{(m,n)}(\xi) = K_N^{(m)}(\psi_\varepsilon^{(m+1,n)}(\mathbf{x}, \tau_\varepsilon^{(n)})), \quad \xi_3 \leqslant 0$$

Proof. As $k\to 0$, the series

$$\psi_\varepsilon^{(0,n)}(\mathbf{x},k) \to R_0^{(0,n,0)}\psi_0^2, \quad \psi_\varepsilon^{(2,n)}(\mathbf{x},k) \to 0, \quad \psi_\varepsilon^{(1,n)}(\mathbf{x},k) \to (R_0^{(1,n,0)} - R_0^{(0,n,0)})\psi_1^2$$

The conditions assumed for the eigenfunctions $\Phi_\varepsilon^{(n)}$ to converge to linear combinations of eigenfunctions ψ_j, normalized in $L_2(\Omega_0 \cup \Omega_1)$, gives Eq. (2.1) in the constants $R_0^{(n,n,0)}$.

By definition, the function Y has the following asymptotic forms as $\rho\to\infty$

$$Y(\xi;\omega^{(m)}) = 1 - \frac{1}{2}c(\omega^{(m)})\rho^{-1} + \sum_{i=1}^{\infty} Z_i^{(m)}(\xi)\rho^{-2i-1},\ \xi_3 \geq 0$$

$$Y(\xi;\omega^{(m)}) = \frac{1}{2}c(\omega^{(m)})\rho^{-1} = \sum_{i=1}^{\infty} Z_i^{(m)}(\xi)\rho^{-2i-1},\ \xi_3 \leq 0$$

We can determine these functions using (2.5) and (2.6) from the condition for the non-decreasing terms of the asymptotic forms to agree on the infinities of the functions $\upsilon_0^{(m,n)}$ and the series $\upsilon_0^{(m,j,n)}$ (see Lemma 2). Further, by equating the coefficients of ρ^{-1} in the asymptotic forms of the functions $\upsilon_0^{(m,n)}$ and the series $V_0^{(m,j,n)}$ we obtain Eqs (2.2) and (2.3). By solving system (2.1)–(2.3) using Lemma 1, we can find the values of the constants $R_0^{(m,n,0)}$ and $\tau_1^{(n)}$. Equating the coefficients of the asymptotic forms of the functions $\upsilon_0^{(m,n)}$ and the series $V_0^{(m,j,n)}$ for the remaining powers of ρ we obtain the polynomials $P_1^{(m,i,n,0)}(D_y)$. As a result of using this matching procedure one obtains that equalities (3.8) hold for $N = 0$.

At the next step, using Lemma 2, we obtain in a similar way that

$$\tilde{V}_1^{(m,m+s,n)} \equiv 0 \to \upsilon_1^{(m,n)} = 0 \to \tau_2^{(n)} = P_i^{(m,n,2i+1,j)} = 0 \tag{3.9}$$

The further proof is carried out by induction using the assertions of Lemmas 2 and 3 (see, for example, [5, 6, 8, 19]).

We will show that formula (2.4) for $\operatorname{Im}\tau_4^{(n)}$ holds (it is easy to establish that $\tau_3^{(n)}$ is real). Boundary-value problem (3.6) for $\operatorname{Im}\upsilon_3^{(m,n)}$ has the form

$$\Delta_\xi \operatorname{Im}\upsilon_3^{(m,n)} = 0,\ \xi \notin \overline{\gamma^{(m)}},\ \partial \operatorname{Im}\upsilon_3^{(m,n)} / \partial\xi_3 = 0,\ \xi \in \gamma^{(m)} \tag{3.10}$$

By (3.9), (3.10) and Lemma 2 we obtain

$$\operatorname{Im}\ \tilde{V}_3^{(1,2,n)} = (\tau_1^{(n)})^3 R_0^{(1,n,0)}\sigma,\ \operatorname{Im}\ \tilde{V}_3^{(m,m,n)} = \operatorname{Im}\ \tilde{V}_3^{(0,1,n)} = 0 \Rightarrow$$

$$\Rightarrow \operatorname{Im}\ \upsilon_3^{(1,n)} = (\tau_1^{(n)})^3 R_0^{(1,n,0)}\sigma Y(\xi_*),\ \operatorname{Im}\upsilon_3^{(0,n)} = 0$$

Equating the coefficients of ρ^{-1} in the asymptotic forms of the functions $\operatorname{Im}\upsilon_3^{(1,n)}$ and the series $V_3^{(1,2,n)}$ and taking into account the asymptotic form of the previously defined function $V_0^{(1,n)}$ we obtain the equations

$$-\tfrac{1}{2}(\tau_1^{(n)})^3 R_0^{(1,n,0)}\sigma c(\omega^{(1)}) = (\tau_1^{(n)})^{-1}\operatorname{Im}\tau_4^{(n)} R_0^{(1,n,0)} - R_0^{(1,n,0)})\psi_1^2 c(\omega^{(1)})$$

Using (2.3) we obtain (2.4) from the last equality. This proves the theorem.

The justification for the asymptotic expansions of the poles $\tau_\varepsilon^{(n)}$ and of the corresponding eigenfunctions $\psi_\varepsilon^{(n)}$ (the proof of Theorem 1) follows from [6, 7]. Formulae (1.4)–(1.6) follow from the asymptotic forms of $\tau_\varepsilon^{(n)}$ and $\psi_\varepsilon^{(n)}$.

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